§ Inner product space.

Goal: Introduce an inner product over V.

to discuss length, distance, orthogonality of vectors in V

$\rightsquigarrow$ more geometric applications

From now on, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Def: Let V vector space over F. An inner product on V is a map

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow F \quad \text{s.t. } \forall \vec{x}, \vec{y}, \vec{z} \in V, \ \forall c \in F.$$

(i) $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

(ii) $\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$.

(iii) $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$ Note: For $F = \mathbb{R}$, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

(iv) $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x} \neq 0$. Note: For $F = \mathbb{C}$, it implies $\langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$

Examples: • For $\vec{x} = (x_1, \cdots, x_n)$, $\vec{y} = (y_1, \cdots, y_n) \in F^n$

We have the standard inner product $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i \overline{y_i}$

When $F = \mathbb{R}$, $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i = \vec{x} \cdot \vec{y}$ "dot product"

• $V = C([a,b])$ be the vector space of real-valued continuous functions on $[a,b]$.

For $f, g \in V$. define $\langle f, g \rangle := \int_a^b f(t) g(t)\, dt$.

Check: this is an inner product on V (i), (ii), (iii) obvious.

(iv). $\forall f \neq 0$. $\langle f, f \rangle = \int_a^b f^2(t)\, dt > 0$.

Since $f^2 > 0$. on some subinterval of $[a,b]$.

$\cong F^{n^2}$

- Let $V = M_{n\times n}(F)$. For $A, B \in V$, define

$$\langle A, B\rangle := \operatorname{tr}(B^* \cdot A) = \sum_{i,j=1}^{n} A_{ij} \cdot \overline{B_{ij}}$$

where B^* is the conjugate transpose / adjoint of B.

$B^* := \overline{B^T}$. More explicitly, if

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \cdot & \cdot & \cdot \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} \rightsquigarrow B^T = \begin{pmatrix} B_{11} & \cdots & B_{n1} \\ \cdot & \cdot & \cdot \\ B_{1n} & \cdots & B_{nn} \end{pmatrix} \rightsquigarrow B^* = \overline{B^T} = \begin{pmatrix} \overline{B_{11}}, & \cdots & \overline{B_{n1}} \\ \cdot & \cdot & \cdot \\ \overline{B_{1n}} & \cdots & \overline{B_{nn}} \end{pmatrix}$$

$$\operatorname{tr}(B^* \cdot A) = \operatorname{tr} \begin{pmatrix} \overline{B_{11}} & \cdots & \overline{B_{n1}} \\ \cdot & \cdot & \cdot \\ \overline{B_{1n}} & \cdots & \overline{B_{nn}} \end{pmatrix} \cdot \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdot & \cdot & \cdot \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} = \cdots = \sum_{i,j=1}^{n} A_{ij} \cdot \overline{B_{ij}}$$

Check: this is an inner product.

- Check (i) ~ (iv) directly (textbook)
- Alternatively, notice $M_{n\times n}(F) \cong F^{n^2}$, $\langle A, B\rangle$ = standard inner product on F^{n^2}

A vector space V equipped with an inner product is called an inner product space.

- If $F=\mathbb{C}$, complex inner product space.
- If $F=\mathbb{R}$. real inner product space.

Prop: Let V be an inner product space. Then $\forall \vec{x}, \vec{y}, \vec{z} \in V$ and $\forall c \in F$.

(a). $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.

(b). $\langle \vec{x}, c\vec{y} \rangle = \bar{c} \cdot \langle \vec{x}, \vec{y} \rangle$.

(c). $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$.

(d). $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$.

(e). If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ $\forall \vec{x} \in V$. then $\vec{y} = \vec{z}$

Pf: (a). $\langle \vec{x}, \vec{y}+\vec{z} \rangle \overset{(iii)}{=} \overline{\langle \vec{y}+\vec{z}, \vec{x} \rangle} \overset{(i)}{=} \overline{\langle \vec{y}, \vec{x} \rangle} + \overline{\langle \vec{z}, \vec{x} \rangle}$

$\overset{(iii)}{=} \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

(b). $\langle \vec{x}, c\vec{y} \rangle \overset{(iii)}{=} \overline{\langle c\vec{y}, \vec{x} \rangle} \overset{(ii)}{=} \bar{c} \cdot \overline{\langle \vec{y}, \vec{x} \rangle} \overset{(iii)}{=} \bar{c} \cdot \langle \vec{x}, \vec{y} \rangle$.

(c). $\langle \vec{x}, \vec{0} \rangle = \langle \vec{x}, \vec{0}+\vec{0} \rangle \overset{(a)}{=} \langle \vec{x}, \vec{0} \rangle + \langle \vec{x}, \vec{0} \rangle$

So $\langle \vec{x}, \vec{0} \rangle = 0$.

(d) follows from (c) and (iv).

(e). If $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle \ \forall \vec{x} \in V$. then $\langle \vec{x}, \vec{y}-\vec{z} \rangle = 0 \ \forall \vec{x} \in V$.

In particular. $\langle \vec{y}-\vec{z}, \vec{y}-\vec{z} \rangle = 0$. $\overset{(d)}{\Rightarrow} \vec{y}-\vec{z} = 0$. $\square$

Def: Let V be an inner product space. For $\vec{x} \in V$.

We define the norm / length of $\vec{x}$. by $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

Prop: $\forall \vec{x}, \vec{y} \in V$ and $\forall c \in F$. We have

(a). $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

(b). $\|\vec{x}\| \geq 0$. and $\|\vec{x}\| = 0$. if $\vec{x} = \vec{0}$.

(c). $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \cdot \|\vec{y}\|$. (Cauchy-Schwarz Inequality)

(d). $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (Triangle Inequality).

pf: (a). $\|c\vec{x}\| = \sqrt{\langle cx, cx\rangle} = \sqrt{\overbrace{c\cdot\bar{c}}^{|c|^2}\langle \vec{x}, \vec{x}\rangle} = |c|\cdot\|\vec{x}\|$

(b). $\|\vec{x}\| \geq 0$ by definition, and
$\|\vec{x}\| = 0$ iff $\langle \vec{x}, \vec{x}\rangle = 0$. iff $\vec{x} = \vec{0}$.

(c) If $\vec{y} = \vec{0}$. then the ineq. certainly holds.
So assume $\vec{y} \neq \vec{0}$.
Note that $\forall c \in F$.

$$0 \leq \|\vec{x} - c\vec{y}\|^2 = \langle \vec{x} - c\vec{y}, \vec{x} - c\vec{y}\rangle = \langle \vec{x}, \vec{x} - c\vec{y}\rangle - c\langle \vec{y}, \vec{x} - c\vec{y}\rangle$$
$$= \langle \vec{x}, \vec{x}\rangle - \bar{c}\langle \vec{x}, \vec{y}\rangle - c\langle \vec{y}, \vec{x}\rangle + c\cdot\bar{c}\langle \vec{y}, \vec{y}\rangle$$

In particular, setting $c = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}$

This gives $$0 \le \langle \vec{x}, \vec{x} \rangle - \cancel{\frac{\overline{\langle \vec{x}, \vec{y} \rangle} \cdot \langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}} - \frac{\langle \vec{x}, \vec{y} \rangle \cdot \langle \vec{y}, \vec{x} \rangle}{\langle \vec{y}, \vec{y} \rangle}$$

$$+ \cancel{\frac{\langle \vec{x}, \vec{y} \rangle \cdot \overline{\langle \vec{x}, \vec{y} \rangle}}{\langle \vec{y}, \vec{y} \rangle}}$$

$$= \|\vec{x}\|^2 - \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2}$$

$$\Rightarrow \quad |\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \cdot \|\vec{y}\|$$

(d). $$\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \overbrace{\langle \vec{y}, \vec{x} \rangle}^{\overline{\langle \vec{x}, \vec{y} \rangle}} + \langle \vec{y}, \vec{y} \rangle .$$

$$= \|\vec{x}\|^2 + 2 \cdot \mathrm{Re} \langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2$$

$$\leq \|\vec{x}\|^2 + 2 |\langle \vec{x}, \vec{y} \rangle| + \|\vec{y}\|^2$$

(Cauchy-Schwartz) $$\leq \|\vec{x}\|^2 + 2 \|\vec{x}\| \cdot \|\vec{y}\| + \|\vec{y}\|^2$$

$$= (\|\vec{x}\| + \|\vec{y}\|)^2$$

$\square$

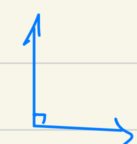
§ Orthogonality

Def: Let V be an inner product space with $\langle \cdot , \cdot \rangle$.

(i). $\vec{x}$ and $\vec{y}$ in V are orthogonal/perpendicular if $\langle \vec{x}, \vec{y} \rangle = 0$.

(ii). A subset S of V is orthogonal if any two distinct vectors in S are orthogonal.

(iii) $\vec{x} \in V$ is a unit vector if $\|\vec{x}\| = 1$.
A subset S of V is orthonormal if S is orthogonal & consists entirely of unit vectors



Note: • Let $S = \{ \vec{v_1}, \vec{v_2}, \cdots \}$. Then

S is orthonormal iff $\langle \vec{v_i}, \vec{v_j} \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

• Let $\vec{v} \in S$ non zero vector, then $\|\vec{v}\| > 0$.

and $\dfrac{\vec{v}}{\|\vec{v}\|} \in V$ is a unit vector.

Such process is called normalization

Example: Let H be the space of continuous complex-valued f^n on $[0, 2\pi]$

$$\langle f, g \rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt \quad \text{for } f, g \in H.$$

Let $S = \{ e^{int} : n \in \mathbb{Z} \} \subset H$.

Then $\langle e^{imt}, e^{int} \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{imt} \cdot \overline{e^{int}}\,dt$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt = \begin{cases} \text{if } m=n. \text{ then } = \frac{1}{2\pi}\int_0^{2\pi} dt = 1 \\ \text{if } m\neq n, \text{ then } = \frac{1}{2\pi i}\frac{1}{m-n} e^{i(m-n)t}\Big|_0^{2\pi} = 0. \end{cases} = \delta_{mn}$$

$\Rightarrow$ S is an orthonormal subset of H.

Why is orthogonal / orthonormal set good?

Prop: Let V be an inner product space and $S = \{\vec{v_1}, \cdots, \vec{v_k}\}$ be an orthogonal subset of V consisting of non-zero vectors.

Then, $\forall \vec{y} \in \mathrm{Span}(S)$, $$\vec{y} = \sum_{i=1}^{k} \frac{\langle \vec{y}, \vec{v_i} \rangle}{\|\vec{v_i}\|^2} \cdot \vec{v_i}.$$

Pf: Write $\vec{y} = \sum_{i=1}^{k} a_i \vec{v_i}$ for some $a_1, \cdots, a_k \in F$.

Taking inner product with $\vec{v_j}$ on both sides gives
$$\langle \vec{y}, \vec{v_j} \rangle = \sum_{i=1}^{n} a_i \langle \vec{v_i}, \vec{v_j} \rangle = a_j \|\vec{v_j}\|^2$$ $\square$

Cor 1. S as above is lin. indep.

Pf: If $\sum_{i=1}^{k} a_i \vec{v_i} = 0$. then $a_i = \frac{\langle 0, \vec{v_i} \rangle}{\|v_i\|^2} = 0$.

Cor 2: If, in addition to above, S is orthonormal,

then $\forall \vec{y} \in \operatorname{Span} S$. $\boxed{\vec{y} = \sum_{i=1}^{k} \langle \vec{y}, \vec{v_i} \rangle \cdot \vec{v_i}}$.

Def: A subset of V is an orthonormal basis for V

if it is a basis which is orthonormal.

Theorem: Suppose $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for V, $T \in \mathcal{L}(V)$

Let $A := [T]_\beta$. then $\boxed{A_{ij} = \langle T(\vec{v}_j), \vec{v}_i \rangle .}$

Pf:

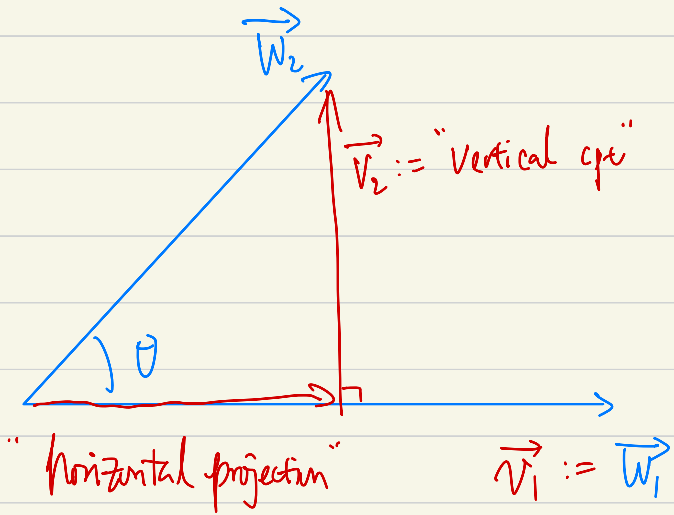
$$A = \begin{array}{c} \\ i^{th}\text{ row} \\ \\ \end{array} \begin{bmatrix} & | & \\ \cdots & [A(\vec{v}_j)]_\beta & \cdots \\ & | & \end{bmatrix}$$

j^{th} column

$$T(\vec{v}_j) = \sum_{i=1}^{n} \underbrace{\langle T(\vec{v}_j), \vec{v}_i \rangle}_{\substack{\| \\ i^{th}\text{ coord of } T(v_j) \\ \| \\ A_{ij}}} \cdot \vec{v}_i$$

$\square$

Next Goal: Obtain an orthogonal / orthonormal basis.



$$\text{length} = \|\vec{w_2}\| \cdot \cos\theta = \frac{\langle \vec{w_2}, \vec{v_1} \rangle}{\|\vec{v_1}\|}$$

Theorem: Let $S_n = \{\vec{w_1}, \dots, \vec{w_n}\}$ be a lin indep subset of inner product space V.

Define $S_n' = \{\vec{v_1}, \dots, \vec{v_n}\}$ by

Gram-Schmidt process

$$\begin{cases} \vec{v_1} = \vec{w_1} \\ \vec{v_2} = \vec{w_2} - \dfrac{\langle \vec{w_2}, \vec{v_1} \rangle}{\|\vec{v_1}\|^2} \vec{v_1} \\ \dots \\ \vec{v_n} = \vec{w_n} - \dfrac{\langle \vec{w_n}, \vec{v_1} \rangle}{\|\vec{v_1}\|^2} \vec{v_1} - \dfrac{\langle \vec{w_n}, \vec{v_2} \rangle}{\|\vec{v_2}\|^2} \cdot \vec{v_2} - \dots - \dfrac{\langle \vec{w_n}, \vec{v_{n-1}} \rangle}{\|\vec{v_{n-1}}\|^2} \cdot \vec{v_{n-1}} \end{cases}$$

$$\frac{\langle \vec{w_2}, \vec{v_1} \rangle}{\|\vec{v_1}\|^2}\vec{v_1} = \underbrace{\frac{\langle \vec{w_2}, \vec{v_1} \rangle}{\|\vec{v_1}\|}}_{\text{length}} \cdot \underbrace{\frac{\vec{v_1}}{\|\vec{v_1}\|}}_{\text{direction}} = \text{"}\mathrm{Proj}_{\vec{v_1}} \vec{w_2}\text{"}$$

Then S_n' is orthogonal (1), s.t. $\mathrm{span}(S_n') = \mathrm{span}(S_n)$ (2)

证: We prove by induction on n.

· For $n=1$, $S_1' = S_1$ so the statement is trivially true.

· Suppose the statement is true for $n = m-1$ i.e., $S_{m-1}' = \{\vec{v_1}, \cdots, \vec{v_{m-1}}\}$ is orthogonal and $\text{span}(S_{m-1}') = \text{span}(S_{m-1})$.

Then for $\vec{v_m} = \vec{w_m} - \sum_{j=1}^{m-1} \frac{\langle \vec{w_m}, \vec{v_j} \rangle}{\|\vec{v_j}\|^2} \cdot \vec{v_j}$,

then $\langle \vec{v_m}, \vec{v_i} \rangle = \langle \vec{w_m}, \vec{v_i} \rangle - \sum_{j=1}^{m-1} \frac{\langle \vec{w_m}, \vec{v_j} \rangle}{\|\vec{v_j}\|^2} \cdot \langle \vec{v_j}, \vec{v_i} \rangle = 0$.

$$= \begin{cases} 0 & \text{if } j \neq i \\ \|v_i\|^2 & \text{if } j = i \end{cases}$$

$\Rightarrow S_m'$ is orthogonal.

It remains to show that $\operatorname{span}(S_m') = \operatorname{span}(S_m)$

$$\underset{S_{m-1}' \cup \{\vec{v}_m\}}{\parallel} \qquad\qquad \underset{S_{m-1} \cup \{\vec{w}_m\}}{\parallel}$$

As $\operatorname{span}(S_{m-1}) = \operatorname{span}(S_{m-1}')$

and $\vec{w}_m = \vec{v}_m + \sum_{i=1}^{m-1} \frac{\langle \vec{w}_m, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i \in \operatorname{span}(S_m')$

$\Rightarrow$ $\operatorname{span}(S_m) \subset \operatorname{span}(S_m')$

Similarly, $\vec{v}_m = \vec{w}_m - \vec{v} \in \operatorname{Span} S_m$

$\vec{v} \in \operatorname{span} S_{m-1}' = \operatorname{span} S_{m-1}$

$\Rightarrow$ $\operatorname{span}(S_m') \subset \operatorname{span}(S_m)$. $\square$

Example: Consider $V = P(\mathbb{R})$ equipped with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

Let $\beta = \{1, x, x^2, \cdots\}$ be the standard basis for $P(\mathbb{R})$

$\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3 \quad \cdots$

- Given $\vec{w}_1 = 1$. G-S process: $\vec{v}_1 = \vec{w}_1 = 1$

- Given $\vec{w}_2 = x$, then $\vec{v}_2 = \vec{w}_2 - \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \cdot \vec{v}_1 = \vec{w}_2 = x$.

Compute $\|\vec{v}_1\|^2 = \int_{-1}^{1} 1 \cdot dx = 2$,

$$\langle \vec{w}_2, \vec{v}_1 \rangle = \int_{1}^{1} x\, dx = 0$$

- Take $\vec{w_3} = x^2$. then $\vec{v_3} = \vec{w_3} - \frac{\langle \vec{w_3}, \vec{v_1} \rangle}{\|\vec{v_1}\|^2} \cdot \vec{v_1} - \frac{\langle \vec{w_3}, \vec{v_2} \rangle}{\|\vec{v_2}\|^2} \cdot \vec{v_2}$.

Compute: $\langle \vec{w_3}, \vec{v_1} \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$. $\|\vec{v_1}\|^2 = 2$.

$\langle \vec{w_3}, \vec{v_2} \rangle = \int_{-1}^{1} x^3 \, dx = 0$. $\|\vec{v_2}\|^2 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$.

Hence $\vec{v_3} = x^2 - \frac{1}{3}$.

- Take $\vec{w_4} = x^3$, $\vec{v_4} = \cdots = x^3 - \frac{3}{5} x$.

Legendre polynomial

And so on

This produces an orthogonal basis $\{\vec{v_1} = 1, \vec{v_2} = x, \vec{v_3} = x^2 - \frac{1}{3}, \cdots\}$ for $P(\mathbb{R})$

$$\text{Orthogonal basis} \longmapsto \text{Orthonormal basis}$$

$$\vec{v_i} \xrightarrow{\text{normalization}} \frac{\vec{v_i}}{\|\vec{v_i}\|},$$

Example: Orthogonal basis $\left\{ 1,\ x,\ x^2 - \frac{1}{3},\ \dots \right\}$ for $P(\mathbb{R})$

Can obtain Orthonormal basis $\left\{ \frac{1}{\sqrt{2}},\ \frac{x}{\sqrt{\frac{2}{3}}},\ \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}},\ \dots \right\}$

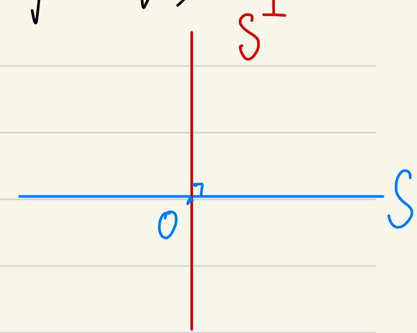
$$\frac{1}{\sqrt{2}} = \frac{\vec{v_1}}{\|\vec{v_1}\|}, \quad \frac{x}{\sqrt{\frac{2}{3}}} = \frac{\vec{v_2}}{\|\vec{v_2}\|}, \quad \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \frac{\vec{v_3}}{\|\vec{v_3}\|}$$

§ Orthogonal Complement

Def: Let S be a nonempty subset of an inner product space V.

The orthogonal complement of S is defined as

$$S^{\perp} := \{\vec{v} \in V ; \langle \vec{v}, \vec{x} \rangle = 0 \quad \forall \vec{x} \in S\}$$



Example:

- $\{0\}^{\perp} = V$. $\qquad V^{\perp} = \{0\}$
- $V = \mathbb{R}^3$. $S = \{\vec{e_3}\}$, then $S^{\perp} = xy\text{-plane} = \operatorname{span}\{\vec{e_1}, \vec{e_2}\}$.

Prop: W subspace, then $W \cap W^{\perp} = \{0\}$

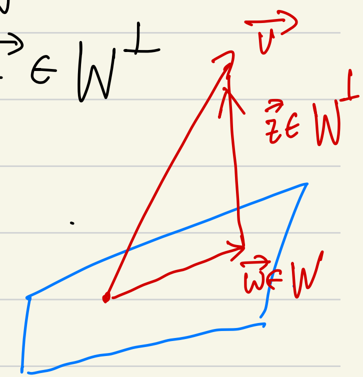
pf: Suppose $\vec{v} \in W \cap W^{\perp}$, then $\langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = 0$ □

Theorem: Let V be an inner product space and $W \subset V$ a finite-dim subspace.

Then $\forall \vec{v} \in V$, there exist unique $\vec{w} \in W$ and $\vec{z} \in W^{\perp}$

s.t. $\boxed{\vec{v} = \vec{w} + \vec{z}}$

orthogonal projection



Furthurmore, if $\{\vec{v}_1, \dots, \vec{v}_k\}$ orthonormal basis for W

then $$\vec{v} = \sum_{i=1}^{k} \langle \vec{v}, \vec{v}_i \rangle \vec{v}_i$$

Pf: · Existence: Given $\vec{v} \in V$. let $\vec{w} := \sum_{i=1}^{k} \langle \vec{v}, \vec{v_i} \rangle \vec{v_i}$, $\vec{z} := \vec{v} - \vec{w}$.

Check $\vec{z} \in W^{\perp}$.

Compute $\langle \vec{z}, \vec{v_j} \rangle = \langle \vec{v}, \vec{v_j} \rangle - \sum_{i=1}^{k} \langle \vec{v}, \vec{v_i} \rangle \langle \vec{v_i}, \vec{v_j} \rangle = 0$, $\forall j$

$\Rightarrow \vec{z} \in W^{\perp}$.

· Uniqueness: Suppose $\exists\ \vec{w}' \in W$ and $\vec{z}' \in W^{\perp}$ s.t. $\vec{w}' + \vec{z}' = \vec{v} = \vec{w} + \vec{z}$

Then $\vec{w} - \vec{w}' = \vec{z}' - \vec{z} \in W \cap W^{\perp} = \{0\}$.

Thus $\vec{w} = \vec{w}'$ and $\vec{z} = \vec{z}'$. $\square$

Theorem: Suppose $S = \{\vec{v}_1, \cdots, \vec{v}_k\}$ is an orthonormal set in an n-dimension inner product space V. Then

(a). S can be extended to an orthonormal basis $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ for V.

(b). If $W = \text{Span}(S)$. then $S_1 = \{\vec{v}_{k+1}, \cdots, \vec{v}_n\}$ is an orthonormal basis for $W^\perp$.

(c) If W is any subspace of V. then $\dim V = \dim W + \dim W^\perp$

Pf: (a). Given $S = \{\vec{v_1}, \cdots, \vec{v_k}\}$ orthonormal, then it is lin indep.

$\xrightarrow{\text{extension}} \{\vec{v_1}, \cdots, \vec{v_k}, \vec{w_{k+1}}, \cdots, \vec{w_n}\}$. basis for V.

$\xrightarrow{\text{G-S process}} \{\underbrace{\vec{v_1}, \cdots, \vec{v_k}}, \vec{w_{k+1}'}, \cdots, \vec{w_n'}\}$ orthogonal basis

Check: S is orthonormal, $\vec{v_1}, \cdots, \vec{v_k}$ are unchanged in the process.

$\xrightarrow{\text{Normalization}} \{\vec{v_1}, \cdots, \vec{v_k}, \vec{v_{k+1}}, \cdots, \vec{v_n}\}$ orthonormal basis.

(b). Note: $S_1 \subset W^{\perp}$, and S_1 lin. indep.

So it suffices to show $\text{span}(S_1) = W^{\perp}$.

Recall $\vec{v} = \sum_{i=1}^{n} \langle \vec{v}, \vec{v_i} \rangle \cdot \vec{v_i} \quad \forall \vec{v} \in V$

In particular, if $\vec{v} \in W^\perp$, $\langle \vec{v}, \vec{v_i} \rangle = 0$ for $i = 1, \cdots, k$

$$\Rightarrow \vec{v} = \sum_{i=k+1}^{n} \langle \vec{v}, \vec{v_i} \rangle \vec{v_i} \in \operatorname{span}(S_1).$$

(c). By choosing an orthonormal basis $\{\vec{v_1}, \cdots, \vec{v_k}\}$ for W.

and by (a) and (b). we have

$$\dim V = n = k + (n-k) = \dim W + \dim W^\perp \qquad \square$$